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CHARACTERIZATION OF INCIDENCE ALGEBRAS*

Robert B. FEINBERG
Mathematics Department, Clarkson College of Technology, Potsdam, NY 13676, U.S.A.

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We develop an internal characterization of the incidence algebra of a lower finite quasi-ordered set and dualize it to obtain a characterization of the corresponding coalgebra. The algebra characterization consists of two axioms concerning the homomorphic images of the algebra and another axiom stipulating the existence of a faithful module over the algebra which has a distributive lattice of submodules.

0. Introduction

Let Q be a *locally finite quasi-ordered set*, i.e., Q has a relation r which is reflexive and transitive and for which every segment $[x, y] = \{z \in Q;\ x r z r y\}$ is finite. The *incidence algebra* $I(Q)$ of Q over a field K is the algebra of functions $f: Q \times Q \to K$, with the property that $f(x, y) \neq 0 \implies x r y$, under the product $f * g(x, y) = \sum_{xrzry} f(x, z) g(z, y)$. $I(Q)$ is an associative unital topological algebra, under a "standard" topology defined in terms of pointwise convergence of functions. It is also dual to an *incidence coalgebra* structure. Rota [4] has developed the incidence algebra as a fundamental structure of enumerative combinatorial theory.

A quasi-ordered set Q is *lower finite* if for each $x \in Q$, $\{w \in Q, w r x\}$ is finite. The main result of this paper is a characterization of incidence algebras of lower finite quasi-ordered sets.

Theorem 3.4. *Let A be an associative unital complete topological algebra over a field K, where K has the discrete topology. Then $A \approx I(Q)$, Q a lower finite quasi-ordered set, when $I(Q)$ has the standard topology $\Longleftrightarrow$.*

(i) *A has a faithful unital left module M with a distributive lattice of submodules. Further, every finitely generated submodule of M is finite dimensional and A has the*

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coarsest topology such that its action on M is continuous in A, when M has the discrete topology.

(ii) *For every maximal closed ideal J, $A/J \approx K_n$, for some integer n.*

(iii) *For every closed ideal J, the center of A/J is isomorphic to the direct product of copies of K.*

The plan of this paper is as follows. Section 1 is devoted to some elementary results on the structure of incidence algebras and coalgebras. Section 2 is a detailed study of the structure of faithful modules over incidence algebras. Our principal results are in Section 3, where we develop the internal characterization of incidence algebras. We conclude with Section 4, where we dualize the results of Section 3 to obtain a characterization of incidence coalgebras.

We make free use of coalgebra notions here, and refer the reader to [6–9] for more details.

1. Incidence algebras and coalgebras

In this section we will define incidence algebras and coalgebras of quasi-ordered sets and derive some of their basic properties.

Let Q be a non-empty set. A *quasi-order* on Q is a relation $r \subseteq Q \times Q$ which is reflexive and transitive. A *partial order* on Q is a relation $\leq$ on Q which is reflexive, anti-symmetric, and transitive.

An *order ideal* of Q, Q quasi-ordered, is a subset R of Q such that for all $x \in R$, $y \in Q$, if $y\,r\,x$, then $y \in R$.

If Q is a q.o. set, we define an equivalence relation $\sim$ on Q by $x \sim y$ iff $x\,r\,y$ and $y\,r\,x$. For $x \in Q$, let $\bar{x}$ be its $\sim$ equivalence class. Then $\hat{Q}$, the collection of equivalence classes of Q, is a p.o. set under the relation $\leq$ defined by $\bar{x} \leq \bar{y}$ iff $x\,r\,y$. We shall call $\hat{Q}$ the *reduced partially ordered set* corresponding to Q. Note that there is a bijective correspondence between order ideals of Q and order ideals of $\hat{Q}$ given by $R \leftrightarrow \hat{R} = \{\bar{x} \in \hat{Q};\ x \in R\}$, where R is an order ideal of Q.

Szpilrajn's lemma asserts that if P is a p.o. set, then we can label its elements by a totally ordered set. A corollary is that if Q is a q.o. set, then we can label its elements so that for $x_i, x_j \in Q$, $x_i\,r\,x_j \implies i \leq j$ or $x_i \sim x_j$.

If Q is a q.o. set and $x\,r\,y$ in Q, then the *segment* $[x, y]$ is the set $\{z \in Q;\ x\,r\,z\,r\,y\}$. Q is *locally finite* if every segment is finite. It is easy to show that Q is locally finite iff every $\sim$ equivalence class of Q is finite and $\hat{Q}$ is locally finite.

Henceforth, all q.o. sets will be locally finite. In the remainder of the chapter, Q will denote an arbitrary locally finite q.o. set.

Notation. Let $\mathrm{SEG}(Q) = \{(x, y) \in Q \times Q;\ x\,r\,y\}$. Define a relation q on $\mathrm{SEG}(Q)$ by $(x, y)\,q\,(x', y')$ iff $x'\,r\,x$ and $y\,r\,y'$. It is easy to verify that q is a quasi-order. If Q is a p.o. set, then SEG(Q) may be identified with the collection of segments of Q, and q is then a partial order corresponding to set-theoretic inclusion.

We are now ready to consider the principal objects of our study.

Definition 1.1. The *incidence algebra* of Q over a field K, denoted $I(Q)$, is the (associative) algebra of functions from SEG(Q) to K, under pointwise addition and the multiplication $*$, where for $f, g \in I(Q)$, $f * g(x, y) = \sum_{xrzry} f(x, z)\, g(z, y)$. We shall frequently denote multiplication by juxtaposition. Also, we shall generally omit mention of the field K, and assume it is fixed throughout the discussion.

Let $\delta \in I(Q)$ be defined by

$$\delta(u, v) = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

For $x r y$ in Q, let $\delta_{xy} \in I(Q)$ be defined by

$$\delta_{xy}(u, v) = \begin{cases} 1 & \text{if } x = u,\ y = v, \\ 0 & \text{otherwise.} \end{cases}$$

Set $\delta_{xx} = e_x$, $\forall x \in Q$. Then it is easy to verify that δ is the identity of $I(Q)$, that $\{\delta_{xy};\ (x, y) \in \mathrm{SEG}(Q)\}$ is a linearly independent subset of $I(Q)$ which spans $I(Q)$ when Q is finite, and that $\{e_x;\ x \in Q\}$ is a complete set of primitive orthogonal idempotents for $I(Q)$.

Definition 1.2. The *incidence coalgebra* of Q over a field K is the triple $(C(Q), \Delta, \varepsilon)$, where $C(Q)$ is the K-vector space spanned by SEG(Q), $\Delta : C(Q) \to C(Q) \otimes C(Q)$ is defined by $\Delta(x, y) = \sum_{xrzry} (x, z) \otimes (z, y)$, and $\varepsilon : C(Q) \to K$ is defined by

$$\varepsilon(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

It is readily seen that $C(Q)^* \approx I(Q)$, where $C(Q)^*$ is the algebra dual to $C(Q)$.

We now begin our study of $I(Q)$ and $C(Q)$ with the following fundamental lemma, which is well-known.

Lemma 1.3. *A subspace V of $C(Q)$ is a subcoalgebra* $\Leftrightarrow$ *V is spanned by an order ideal of* SEG(Q).

Notation. For $\bar{x} \in Q$ let $C_{\bar{x}} = \mathrm{SEG}(\bar{x})$, the subspace of $C(Q)$ spanned by $\{(x', x'') \in \mathrm{SEG}(Q);\ \bar{x}' = \bar{x}'' = \bar{x}\}$.

Corollary 1.4. *A subcoalgebra D of $C(Q)$ is simple* $\iff$ $D = C_{\bar{x}}$, *for some* $\bar{x} \in Q$.

It follows easily from the corollary that for Q a q.o. set, the coalgebra $C(Q)$ is pointed iff Q is a p.o. set.

Now let D be a subcoalgebra of $C(Q)$. The natural inclusion map $i : D \to C(Q)$ induces an algebra epimorphism $i^* : C(Q)^* \approx I(Q) \to D^*$ defined by $i^*(f) = f \circ i$, for $f \in I(Q)$. Henceforth we shall write $i^*(f)$ as $\bar{f}$. Note also that ker $i^* = D^\perp$ and that $I(Q)/D^\perp \approx D^*$.

By Lemma 1.3, D is spanned by an order ideal of SEG(Q), which may be described set-theoretically as SEG(Q) $\cap$ D. It follows that any $\bar{f} \in D^*$ is uniquely determined by specifying its values on the elements of SEG(Q) $\cap$ D.

The *standard topology* on D^* is defined by stipulating that a net $\{\bar{f}_\lambda\}_{\lambda \in \theta} \subseteq D^*$ converges to $\bar{f} \in D^*$ iff $\forall (x, y) \in$ SEG(Q) $\cap$ D, $\bar{f}_\lambda(x, y) = \bar{f}(x, y)$ eventually. It is easy to verify that D^* equipped with the above topology is a topological algebra, when the ground field K has the discrete topology, also that this topology coincides with the finite topology (cf. [7]). It then follows from results in [7] that D^* is a complete topological algebra under the standard topology. Henceforth all topological notions in D^* will be relative to this topology. Typically $D = C(Q)$, so that $D^* \approx I(Q)$.

Now let θ be a collection of finite subsets of SEG(Q) $\cap$ D which covers SEG(Q) $\cap$ D and which is directed by inclusion. For each $\lambda \in \theta$, let $\bar{f}_\lambda = \sum_{(x,y) \in \lambda} \bar{f}(x, y)\bar{\delta}_{xy}$, where $\bar{f} \in D^*$. It is shown in [4, Section 3] that $\{\bar{f}_\lambda\}_{\lambda \in \theta} \to \bar{f}$ in the standard topology on D^*. This allows us to consider every element of D^* as a formal linear combination of possibly infinitely many elements of the set $\{\bar{\delta}_{xy};$ $(x, y) \in$ SEG(Q) $\cap$ $D\}$.

We now state some properties of the closed ideals of $I(Q)$.

Lemma 1.5 (cf. [4, Section 3]). $J \lhd I(Q)$ *is a maximal closed ideal* $\iff$ $J = C_{\bar{x}}^\perp$ *for some* $\bar{x} \in \hat{Q}$.

Lemma 1.6. *Let* $J = C_{\bar{x}}^\perp$ *be a maximal closed ideal of* $I(Q)$. *Then* $I(Q)/J \approx K_n$, *where* $n = |\bar{x}|$ *and* K_n *is the algebra of* $n \times n$ *matrices over* K.

The lemma to follow generalizes a result in [1] about the center of $I(Q)$, written $\mathbf{Z}[I(Q)]$, for Q a p.o. set. The proof is essentially an elaboration of the proof there.

Lemma 1.7. *Let* J *be a closed ideal of* $I(Q)$. *Then* $\mathbf{Z}[I(Q)/J]$ *is isomorphic to the direct product of copies of the ground field* K.

2. Modules over incidence algebras

In this section we shall obtain some results on faithful modules over incidence algebras needed subsequently for the characterization of these algebras. Unless otherwise noted, all modules considered are unital left modules. All algebras are over a fixed field K.

Definition 2.1. Let A be a topological algebra over the field K, having the discrete topology, and let M be a module over A. Then M is *topologically compatible* if the action of A on M is continuous in A, when M has the discrete topology.

Definition 2.2. Let M be a module over $I(Q)$, Q a locally finite q.o. set. Then M is *good* if $e_x \cdot M$ is 1-dimensional, $\forall x \in Q$.

Definition 2.3. Let Q be a q.o. set. For $x \in Q$, set $L_x = \{u \in Q \text{ such that } urx\}$. Then Q is *lower finite* if for all $x \in Q$, L_x is finite.

Lemma 2.4. *Let Q be a lower finite quasi-ordered set. Then $I(Q)$ has a faithful, topologically compatible, good module $M(Q)$, whose dimension equals the cardinality of Q.*

Proof. Let $M(Q)$ be the K-vector space with basis $\{x\}_{x\in Q}$. Let $f \in I(Q)$ act on a basis element $x \in M(Q)$ by $f \cdot x = \sum_{yrx} f(y, x)y$, and extend by linearity to obtain an action of f on $M(Q)$. It is readily verified that this action makes $M(Q)$ an $I(Q)$-module with the above properties.

Lemma 2.5. *Let Q be a locally finite quasi-ordered set, and suppose $I(Q)$ has a faithful, topologically compatible, good module M. Then Q is lower finite.*

Proof. For each $z \in Q$, choose $m_z \in e_z \cdot M \setminus \{0\}$. Let $B = \{m_z\}_{z\in Q}$. It is not difficult to show that B is a basis for M. Also, the faithfulness of M implies that for urx in Q, $\delta_{ux} \cdot m_x = a_{ux}m_x$, for some non-zero $a_{ux} \in K$.

Suppose now that $L_x = \{u \in Q \text{ such that } urx\}$ is infinite. Let θ' be a collection of finite subsets of L_x which covers L_x and which is directed by inclusion. Then

$$\{\delta_s\}_{s\in\theta'} \to \sum_{urx} \delta_{ux}$$

in the standard topology, where $\delta_s = \sum_{u\in s} \delta_{ux}$. Hence $\exists T \in \theta'$ such that for all $T' \supseteq T$,

$$\left(\sum_{u\in T'} \delta_{ux}\right) \cdot m_x = \left(\sum_{u\in T} \delta_{ux}\right) \cdot m_x .$$

Note that

$$\left(\sum_{u\in T'} \delta_{ux}\right) \cdot m_x = \sum_{u\in T'} \delta_{ux} \cdot m_x = \sum_{u\in T'} a_{ux}m_u .$$

But since L_x is infinite, $\exists u' \in L_x \setminus T$. Then

$$\left(\sum_{u\in T\cup\{u'\}} \delta_{ux}\right)\cdot m_x = a_{u'x}m_{u'} + \left(\sum_{u\in T} \delta_{ux}\right)\cdot m_x \neq \left(\sum_{u\in T} \delta_{ux}\right)\cdot m_x.$$

From this contradiction we obtain that L_x is always finite. Hence Q is lower finite.

Unless otherwise noted, all quasi-ordered sets Q considered in the remainder of this section will be lower finite.

Now suppose that A is an algebra over the field K with a faithful module M. Let us define the *M-topology* on A by stipulating that a net $\{\alpha_\lambda\}_{\lambda\in\theta}\subseteq A$ converges to $\alpha\in A$ iff $\forall m\in M$, $\alpha_\lambda\cdot m=\alpha\cdot m$ eventually. It is easy to verify that A equipped with the M-topology is a topological algebra when K has the discrete topology. In addition, the M-topology is the coarsest topology on A such that the action of A on M is continuous in A, when M has the discrete topology.

Lemma 2.6. *Let Q be a lower finite q.o. set. Let M be a faithful, topologically compatible, good module over $I(Q)$. Then the standard topology on $I(Q)$ agrees with the M-topology on $I(Q)$.*

Lemma 2.7. *Let Q a lower finite quasi-ordered set. Let M be a faithful, topologically compatible, and good module over $I(Q)$. Choose $m_z\in e_z\cdot M\setminus\{0\}$, for each $z\in Q$, so that $B=\{m_z\}_{z\in Q}$ is a basis for M, as in Lemma 2.5. Then a subspace N of M is a submodule iff there is an order ideal $\hat{P}$ of $\hat{Q}$ such that N is spanned by $\{m_x\in M$ such that $\bar{x}\in\hat{P}\}$.*

Proof. ($\Longleftarrow$) Easy.

($\Longrightarrow$) Suppose N is a submodule of M and that $\sum_{i=1}^n a_i m_{x_i}\in N$, $a_i\neq 0$, $1\leqslant i\leqslant n$. Then for $1\leqslant j\leqslant n$,

$$e_{x_j}\cdot\sum_{i=1}^n a_i m_{x_i} = a_j m_{x_j}\in N.$$

It follows that $m_{x_j}\in N$ and hence that N is spanned by some subset of B. Now let $P=\{x\in Q$ such that $m_x\in N\}$. It follows from the above that $N=\operatorname{span}\{m_x\in M$ such that $x\in P\}$. Note that $x\in P$ and $wrx\Longrightarrow m_x\in N$ and $\delta_{wx}m_x = a_{wx}m_w\in N\Longrightarrow m_w\in N$ (since $a_{wx}\neq 0$) $\Longrightarrow w\in P$. Hence P is an order ideal of Q and $\hat{P}$ is an order ideal of $\hat{Q}$. Finally, $x\in P\Longleftrightarrow\bar{x}\in\hat{P}$, so that $N=\operatorname{span}\{m_x\in M$ such that $\bar{x}\in\hat{P}\}$, showing that N is of the required form.

Notation. Let Q and M be as in Lemma 2.7. Let $\hat{P}$ be an order ideal of $\hat{Q}$. Set $M_{\hat{P}}=\operatorname{span}\{m_x\in M;\ \bar{x}\in\hat{P}\}$. Then $M_{\hat{P}}$ is a submodule of M and every submodule is of this form, by the lemma.

Theorem 2.8. *Let Q be a lower finite quasi-ordered set. Let M be a faithful, topologically compatible module over $I(Q)$. Then M is good $\Longleftrightarrow$ M has a distributive lattice of submodules.*

Proof. ($\Longrightarrow$) Let L_M be the lattice of submodules of M and let $H(\hat{Q})$ be the lattice of order ideals of $\hat{Q}$, which is distributive, as is easily proven. Lemma 2.7 establishes that there is a natural map $\varphi : H(\hat{Q}) \to L_M$ defined by $\varphi(\hat{P}) = M_{\hat{P}}$. It is easily verified that φ is a lattice isomorphism. Therefore L_M is distributive.

($\Longleftarrow$) Suppose $\exists x \in Q$ such that $e_x \cdot M$ is *not* 1-dimensional. Choose linearly independent elements m_1 and m_2 from $e_x \cdot M$. Let $N_1 = I(Q) \cdot m_1$, $N_2 = I(Q) \cdot m_2$, $N_3 = I(Q) \cdot (m_1 + m_2)$.

Now N_1, N_2, N_3 are submodules of M and $m_1 + m_2 \in N_1 + N_2$. Hence $N_3 = I(Q) \cdot (m_1 + m_2) \subseteq N_1 + N_2$. In particular, $(N_1 + N_2) \cap N_3 = N_3$, so that $e_x \cdot [(N_1 + N_2) \cap N_3] = e_x \cdot N_3 = K(m_1 + m_2)$. But $e_x \cdot N_1 = Km_1$, $e_x \cdot N_2 = Km_2$, so that $e_x \cdot (N_1 \cap N_3) = e_x \cdot (N_2 \cap N_3) = 0$. Hence $e_x \cdot [(N_1 \cap N_3) + (N_2 \cap N_3)] = 0$. We conclude that $(N_1 + N_2) \cap N_3 \neq (N_1 \cap N_3) + (N_2 \cap N_3)$, so that the lattice of submodules of M is *not* distributive.

In the sequel we will refer to faithful, topologically compatible, good modules over $I(Q)$ as *faithful distributive* modules over $I(Q)$.

We now specialize our discussion to finite-dimensional incidence algebras.

Lemma 2.9. *Let Q be a finite quasi-ordered set. Then $I(Q)$ has a finite-dimensional faithful unital left module with a distributive lattice of submodules.*

Proof. This follows from Lemma 2.4 and Theorem 2.8.

Lemma 2.10. *Let Q be a finite quasi-ordered set of cardinality n, and let M be a faithful module over $I(Q)$. TFAE (The following are equivalent):*

(1) *M is good,*

(2) $\dim M = n$,

(3) *M has a distributive lattice of submodules.*

Proof. (1) $\Longleftrightarrow$ (2). Since $\{e_x\}_{x \in Q}$ is a collection of orthogonal idempotents adding to the identity, we have that $M = \bigoplus_{x \in Q} e_x \cdot M$ (vector space direct sum), so that $\dim M = \Sigma_{x \in Q} \dim(e_x \cdot M)$. Note that $e_x \cdot M \neq 0$, $\forall x \in Q$, since M is faithful. It then follows that $e_x \cdot M$ is 1-dimensional, $\forall x \in Q \Longleftrightarrow \dim M = |Q| = n$.

(1) $\Longleftrightarrow$ (3) This follows from Theorem 2.8.

It is now convenient to give the following weak characterization of finite-dimensional incidence algebras. The proof follows readily from previous results.

Lemma 2.11. *Let A be a finite-dimensional algebra over the field K. TFAE*

(1) *$A \approx I(Q)$, Q a finite quasi-ordered set of cardinality n.*

(2) *A is isomorphic to a subalgebra of K_n containing $\{e_{ii}\}_{i=1}^n$, where e_{ij} is the matrix unit with 1 in the (i, j)-position and 0's elsewhere.*

(3) *A has a faithful module M of dimension n and a complete set of n orthogonal idempotents* $\{e_i\}_{i=1}^n$.

We now return to consideration of algebras of not necessarily finite dimension.

Definition 2.12. Let M be a module over an algebra A with the property that each of its finitely generated submodules is finite-dimensional. We define a quasi-ordered set O_M (up to its isomorphism class) as follows: the $\sim$ equivalence classes of O_M are in 1–1 correspondence with the finitely generated join-irreducible submodules of M, and if $X_{M_i}, X_{M_j} \in \hat{O}_M$ correspond to the submodules M_i, M_j respectively, then

$$X_{M_i} \leqslant X_{M_j} \quad \text{in} \quad \hat{O}_M \iff M_i \subseteq M_j.$$

Furthermore, the cardinality of the equivalence class X_{M_i} is $\dim M_i - \dim M_i'$, where M_i' is the join of all proper submodules of M_i, and hence the unique maximal submodule of M_i.

Lemma 2.13. *Let M be an A-module with a distributive lattice of submodules such that every finitely-generated submodule of M is finite-dimensional.*

(1) *If M' is an arbitrary submodule of M, then M' is finite-dimensional* $\iff$ *$O_{M'}$ is a finite order ideal of O_M.*

(2) *Every finite order ideal of O_M is of the form $O_{M'}$, M' some finite-dimensional submodule of M.*

Proof. (1) $O_{M'}$ is obviously an order ideal of O_M. Necessity follows from elementary lattice theory [3, Chapter 9] and sufficiency is readily established.

(2) It suffices to show that every finite order ideal of $\hat{O}_M$ is of the form $\hat{O}_{M'}$, M' a finite-dimensional submodule of M. Let $\hat{R}$ be a finite order ideal of $\hat{O}_M$. Then there is a finite collection $\{M_i\}_{i=1}^n$ of submodules of M consisting of all finitely generated join-irreducible submodules N such that $X_N \in \hat{R}$ and N is maximal in M with respect to this property. Set $M' = \sum_{i=1}^n M_i$, so that M' is a finite-dimensional submodule of M. We show that $\hat{R} = \hat{O}_{M'}$. Note first that $X_{N'} \in \hat{R} \implies N' \subseteq M_j$, some $j \in [1, n] \implies N' \subseteq M' \implies X_{N'} \in \hat{O}_{M'}$, so that $\hat{R} \subseteq \hat{O}_{M'}$. Conversely, assume $X_{N'} \in \hat{O}_{M'}$. Then N' is a join-irreducible submodule of M'. Since M has a distributive lattice of submodules, $N' = N' \cap \sum_{i=1}^n M_i = \sum_{i=1}^n N' \cap M_i$. Since N' is join-irreduccible, there is some $j \in [1, n]$ such that $N' = N' \cap M_j$. Hence $N' \subseteq M_j$. It follows that $X_{N'} \leqslant X_{M_j}$ in $\hat{O}_M$, and since X_{M_j} is in the order ideal $\hat{R}$, it follows that $x_{N'} \in \hat{R}$ also. Hence $\hat{O}_{M'} \subseteq \hat{R}$.

Corollary 2.14. *Let M be an A-module, as in Lemma* 2.13. *Then O_M is lower finite.*

We now specialize the above discussion to incidence algebras. Proof of the first result is routine.

Lemma 2.15. *Let M be a faithful distributive module over $I(Q)$, Q a lower finite quasi-ordered set. Then a submodule $M_{\hat{P}}$ of M is join-irreducible (in the lattice of submodules of M) $\Longleftrightarrow$ $\hat{P}$ is cyclic as an order ideal of $\hat{Q}$ (i.e., $\exists \bar{y} \in \hat{Q}$ such that $\hat{P} = \{\bar{x} \leq \bar{y}\}$), and that in this case, $M_{\hat{P}}$ is finitely generated.*

Theorem 2.16. *Let Q be a lower finite quasi-ordered set and M be a faithful distributive module over $I(Q)$.*

(1) *Every finitely generated submodule of M is finite-dimensional.*

(2) $O_M \approx Q$.

Proof. (1) This is readily established.

(2) It follows readily from Lemma 2.15 that the p.o. set of cyclic order ideals of $\hat{Q}$ is isomorphic to the p.o. set of finitely generated join-irreducible submodules of M by the correspondence $\bar{y} \to M_{\bar{y}}$, where we have identified $\bar{y} \in \hat{Q}$ with the cyclic order ideal generated by it. Note also that $\hat{Q}$ is in turn isomorphic to the p.o. set of its cyclic order ideals, by this same identification. Hence $\psi : \hat{Q} \to \hat{O}_M$ defined by $\psi(\bar{y}) = X_{M_{\bar{y}}}$ is a p.o. isomorphism.

Next note that $M'_{\bar{y}}$, the unique maximal submodule of $M_{\bar{y}}$, is equal to $M_{\hat{R}}$, where $\hat{R}$ is the unique maximal order ideal of $\hat{Q}$ contained in the order ideal generated by $\bar{y}$, namely $\{\bar{z} \in \hat{Q};\ \bar{z} < \bar{y}\}$. We then obtain that $\dim M_{\bar{y}} - \dim M'_{\bar{y}} = |\bar{y}|$, so that the cardinality of $X_{M_{\bar{y}}}$ as an equivalence class in O_M is $|\bar{y}|$. Thus $|X_{M_{\bar{y}}}| = |\psi(\bar{y})| = |\bar{y}|$. This establishes that there is a quasi-order isomorphism between Q and O_M, as is readily seen.

An easy corollary is that if Q, Q' are lower finite q.o. sets such that $I(Q)$ and $I(Q')$ are isomorphic as topological algebras, then $Q \approx Q'$.

3. Characterization of incidence algebras

We now have the tools for characterizing incidence algebras of finite q.o. sets. This is the central result of the paper, and it will be applied subsequently to obtain characterizations of incidence algebras and coalgebras of lower finite q.o. sets.

Theorem 3.1. *Let A be a finite-dimensional algebra over a field K. Then $A \approx I(Q)$, Q a finite quasi-ordered set $\Longleftrightarrow$*

(i) *A has a finite-dimensional faithful module M with a distributive lattice of submodules.*

(ii) *Every simple homomorphic image of A is isomorphic to K_n, for some integer n.*

(ii) *The center of every homomorphic image of A is isomorphic to the direct product of copies of K.*

Proof. ($\Longrightarrow$) (i) This is Lemma 2.9. (ii) This follows from Lemma 1.6. (iii) This follows from Lemma 1.7.

($\Longleftarrow$) If M is irreducible, then A is primitive, hence simple, hence isomorphic to K_n, $n = \dim M$, by (ii). Therefore $A \approx I(Q)$, where Q is the q.o. set of size n in which any pair of elements are $\sim$-equivalent. In particular, if $\dim M = 1$, then $A \approx I(Q_1)$, where Q_1 is the quasi-ordered set with a single element. Thus we assume that M is reducible and use induction on $\dim M$.

Choose N a maximal submodule of M. Assume $\dim N = q < \dim M = n$. By induction, $A/\mathrm{Ann}\, N \approx I(Q')$, Q' a finite quasi-ordered set. Further, $|Q'| = q$, by Lemma 2.10. If $\mathrm{Ann}\, N = 0$, we are done. Thus, assume $\mathrm{Ann}\, N \neq 0$.

It follows from results in the last section that N has a basis $B = \{m_i\}_{i=1}^q$ such that $A/\mathrm{Ann}\, N \approx I(Q')$ is represented by its action on B as a subalgebra of K_q containing $\{e_{ii}\}_{i=1}^q$ and spanned by matrix units, with square blocks of matrix units around the main diagonal and all other matrix units occurring in blocks above the diagonal, each at the intersection of the rows occupied by one of the diagonal blocks with the columns occupied by another. Extend B to a basis $B_1 = \{m_i\}_{i=1}^n$ of M. Then every element of A is represented by its action on B_1 as an $n \times n$ matrix with non-zero entries only in the first q rows and last $n-q$ column, $\mathrm{Ann}\, N$ is represented as the matrices in A with 0's in the first q columns, and $\mathrm{Ann}\,(M/N)$ is represented as the matrices in A with 0's in the last $n-q$ rows.

Further, $A/\mathrm{Ann}\, N$ is represented by its action on B as

$$\left\{ \begin{bmatrix} a_{11} & \cdots & a_{1q} \\ \vdots & & \vdots \\ a_{q1} & \cdots & a_{qq} \end{bmatrix} \in K_q \quad \text{such that} \quad \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1q} & \\ \vdots & & \vdots & * \\ a_{q1} & \cdots & a_{qq} & \\ \hline & 0 & & * \end{array}\right] \in A \right\},$$

where $*$ denotes a block of unspecified entries from K, and $A/\mathrm{Ann}\,(M/N)$ is represented by its action on $\{m_i + N\}_{i=q+1}^n$ as

$$\left\{ \begin{bmatrix} a_{q+1,q+1} & \cdots & a_{q+1,n} \\ \vdots & & \vdots \\ a_{n,q+1} & \cdots & a_{nn} \end{bmatrix} \in K_{n-q} \quad \text{such that} \quad \left[\begin{array}{c|ccc} * & & * & \\ \hline & a_{q+1,q+1} & \cdots & a_{q+1,n} \\ 0 & \vdots & & \vdots \\ & a_{n,q+1} & \cdots & a_{nn} \end{array}\right] \in A \right\}.$$

Now (M/N) is irreducible, so that $A/\mathrm{Ann}\,(M/N)$ is primitive, hence simple, hence isomorphic to K_{n-q}. Further, $\mathrm{Ann}\,(M/N)$ is a maximal ideal of A. If $\mathrm{Ann}\,(M/N) = 0$, then $A \approx K_{n-q}$ and we are done. Thus we assume $\mathrm{Ann}\,(M/N) \neq 0$. There are two cases. In each case we will consider the matrix units of $A/\mathrm{Ann}\, N$ and calculate suitably normalized preimages of them in A. We shall then use these preimages to show that A fulfills the hypotheses of Lemma 2.11 and hence is an incidence algebra (Case 1), or else to reach a contradiction (Case 2).

Case 1: $\operatorname{Ann} N \not\subseteq \operatorname{Ann}(M/N)$. Then

$$\operatorname{Ann} N + \operatorname{Ann}(M/N) = A$$

and

$$\operatorname{Ann} N/(\operatorname{Ann}(M/N) \cap \operatorname{Ann} N) \approx A/\operatorname{Ann}(M/N) \approx K_{n-q}.$$

Now $\operatorname{Ann}(M/N) \cap \operatorname{Ann} N$ is represented by its action on B_1 as $n \times n$ matrices in A with 0's in the last $n-q$ rows and first q columns, and $\operatorname{Ann} N/(\operatorname{Ann}(M/N) \cap \operatorname{Ann} N)$ is represented by its action on $\{m_i + N\}_{i=q+1}^{n}$ as

$$\left\{ \begin{bmatrix} a_{q+1,q+1} & \cdots & a_{q+1,n} \\ \vdots & & \vdots \\ a_{n,q+1} & \cdots & a_{nn} \end{bmatrix} \in K_{n-q} \text{ such that } \left[\begin{array}{c|ccc} 0 & & * & \\ \hline & a_{q+1,q+1} & \cdots & a_{q+1,n} \\ 0 & \vdots & & \vdots \\ & a_{n,q+1} & \cdots & a_{nn} \end{array}\right] \in \operatorname{Ann} N \subset A \right\}.$$

In particular,

$$\left\{ \left[\begin{array}{c|c} 0 & * \\ \hline 0 & e_{ij} \end{array}\right] \right\}_{q+1 \le i,j \le n} \subseteq A. \tag{3.1}$$

We also know that

$$\left\{ \left[\begin{array}{c|c} e_{ii} & * \\ \hline 0 & * \end{array}\right] \right\}_{i=1}^{q} \subseteq A,$$

since $A/\operatorname{Ann} N$ is an incidence algebra, and by (3.1) we obtain

$$\left\{ \left[\begin{array}{c|c} e_{ii} & * \\ \hline 0 & 0 \end{array}\right] \right\}_{i=1}^{q} \subseteq A.$$

More explicitly, for $1 \le i \le q$ there is an element

$$\mathbf{e}_{ii} = e_{ii} + \sum_{\substack{1 \le k \le q \\ q+1 \le j \le n}} v_{kj} e_{kj} \in A$$

and for $q+1 \le i \le n$, there is an element

$$\mathbf{e}_{ii} = e_{ii} + \sum_{\substack{1 \le j \le q \\ q+1 \le k \le n}} b_{jk} e_{jk} \in A.$$

Squaring $\mathbf{e}_{ii}$, $1 \le i \le q$, and subtracting

$$\sum_{\substack{q+1 \le j \le n \\ e_{ij} \in A}} v_{ij} e_{ij},$$

we obtain $\hat{e}_{ii} \in A$, where

$$\hat{e}_{ii} = e_{ii} + \sum_{\substack{q+1 \le j \le n \\ e_{ij} \notin A}} v_{ij} e_{ij}.$$

Similarly, for $q+1 \le i \le n$, we obtain $\hat{e}_{ii} \in A$, where

$$\hat{e}_{ii} = e_{ii} + \sum_{\substack{1 \le j \le q \\ e_{ji} \notin A}} b_{ji} e_{ji}.$$

It is easy to verify that $\{\hat{e}_{kk}\}_{k=1}^{n}$ is a complete set of n orthogonal idempotents for A. Since A has a faithful module of dimension n, namely M, we conclude by Lemma 2.11 that $A \approx I(Q)$, Q a quasi-ordered set of cardinality n.

Case 2: $\operatorname{Ann} N \subseteq \operatorname{Ann}(M/N)$. We will derive a contradiction to hypothesis (iii) or to the distributivity of the lattice of submodules of M.

Let us assume that $e_{ij} \in A/\operatorname{Ann} N$ lifts to

$$\left[\begin{array}{c|c} e_{ij} & * \\ \hline 0 & A_{ij} \end{array}\right] \in A.$$

Then the map $\psi : A/\operatorname{Ann} N \to A/\operatorname{Ann}(M/N)$ defined by $\psi(e_{ij}) = A_{ij}$ is the canonical projection of $A/\operatorname{Ann} N$ by $\operatorname{Ann}(M/N)/\operatorname{Ann} N$. Hence ψ is a homomorphism, and $\ker \psi = \operatorname{Ann}(M/N)/\operatorname{Ann} N$. Thus

$$A_{ij} A_{kl} = \begin{cases} A_{il} & \text{if } j = k, \\ 0 & \text{if } j \ne k, \end{cases}$$

and in particular, $A_{ii}^2 = A_{ii}$, $1 \le i \le q$, and $A_{ii} A_{jj} = 0$, $i \ne j$.

Recall from the preliminary discussion that $A/\operatorname{Ann} N \approx I(Q')$, Q' a quasi-ordered set of cardinality q, and that $A/\operatorname{Ann}(M/N) \approx K_{n-q}$. Thus

$$\frac{A/\operatorname{Ann} N}{\operatorname{Ann}(M/N)/\operatorname{Ann} N} \approx \frac{I(Q')}{\ker \psi}$$

and

$$\frac{A/\operatorname{Ann} N}{\operatorname{Ann}(M/N)/\operatorname{Ann} N} \approx \frac{A}{\operatorname{Ann}(M/N)} \simeq K_{n-q}.$$

This shows that $\ker \psi$ is a maximal ideal of $I(Q')$. We then obtain by Lemma 1.5 that $\ker \psi = C_{\bar{x}}^{\perp}$, for some $\bar{x} \in \hat{Q}'$ such that $|\bar{x}| = n-q$.

Let us assume $\{e_u\}_{u\in\mathfrak{X}}$ is represented by its action on B as $\{e_{ii}\}_{i=a+1}^{a+n-q}$, for some integer $a\in[0,2q-n]$. Then $\ker\psi$ is the subspace of $A/\operatorname{Ann}N$ spanned by $\{e_{ij}$ such that $(1\leq i,j\leq q)$ and $(i$ or $j\notin[a+1,\ a+n-q]\}$, and $\psi(e_{ij})=A_{ij}\neq 0$ iff i and $j\in[a+1,\ a+n-q]$. In particular, $\{A_{ij}\}_{a+1\leq i,j\leq a+n-q}$ is a matrix units basis for $A/\operatorname{Ann}(M/N)\approx K_{n-q}$.

Let us choose a basis $\{m'_i+N\}_{i=q+1}^{n}$ for M/N so that $A_{a+j,a+k}\in A/\operatorname{Ann}(M/N)$ is represented as $e_{jk}\in K_{n-q}$, for $1\leq j,k\leq n-q$. Let $B_2=\{m_i\}_{i=1}^{q}\cup\{m'_j\}_{j=q+1}^{n}$. Then B_2 is a basis for M with respect to which $e_{ij}\in A/\operatorname{Ann}N$ lifts to

$$\mathbf{e}_{ij}=e_{ij}+\sum_{\substack{1\leq s\leq q\\ q+1\leq p\leq n}}c(i,j)_{sp}e_{sp}\in A,$$

for i or $j\in[1,a]\cup[a+n-q+1,q]$, and $e_{a+k,a+l}\in A/\operatorname{Ann}N$ lifts to

$$\mathbf{e}_{a+k,a+l}=e_{a+k,a+l}+e_{q+k,q+l}+\sum_{\substack{1\leq s\leq q\\ q+1\leq p\leq n}}c(a+k,a+l)_{sp}e_{sp}\in A,\qquad\text{for } 1\leq k,\ l\leq n-q.$$

Set $c_{ip}=c(i,i)_{ip}$, $i\in[1,a]\cup[a+n-q+1,q]$. For such i, squaring $\mathbf{e}_{ii}$ and subtracting $\sum_{p=q+1,e_{ip}\in A}^{n}c_{ip}e_{ip}$, we obtain $\hat{e}_{ii}\in A$, where

$$\hat{e}_{ii}=e_{ii}+\sum_{\substack{p=q+1\\ e_{ip}\notin A}}^{n}c_{ip}e_{ip}.$$

Set $c_{a+k,p}=c(a+k,a+k)_{a+k,p}$ and $d_{s,q+k}=c(a+k,a+k)_{s,q+k}$, $1\leq k\leq n-q$. Proceeding as above, we obtain $\hat{e}_{a+k,a+k}\in A$, where

$$\hat{e}_{a+k,a+k}=e_{a+k,a+k}+e_{q+k,q+k}+\sum_{\substack{p=q+1\\ p\neq q+k\\ e_{a+k,p}\notin A}}^{n}c_{a+k,p}e_{a+k,p}+\sum_{\substack{s=1\\ s\neq a+k\\ e_{s,q+k}\notin A}}^{q}d_{s,q+k}e_{s,q+k}.$$

By a sequence of multiplications and subtractions we may similarly obtain that for $1\leq i\leq q$, $j\in[1,a]\cup[a+n-q+1,q]$, $i\neq j$, there is an element $\hat{e}_{ij}\in A$, where

$$\hat{e}_{ij}=e_{ij}+\sum_{\substack{p=q+1\\ e_{ip}\notin A}}^{n}c_{jp}e_{ip}.$$

Also, for $i\in[1,a]\cup[a+n-q+1,q]$, $1\leq l\leq n-q$, there is an element $\hat{e}_{i,a+l}\in A$ where

$$\hat{e}_{i,a+l}=e_{i,a+l}+\sum_{\substack{p=q+1\\ e_{ip}\notin A}}^{n}c_{a+l,p}e_{ip}.$$

Finally, for $1 \leq k,\ l \leq n-q,\ k \neq l$, there is an element $\hat{e}_{a+k,a+l} \in A$, where

$$\hat{e}_{a+k,a+l} = e_{a+k,a+l} + e_{q+k,q+l} + \sum_{\substack{p=q+1\\ p\neq q+l\\ e_{a+k,p}\notin A}}^{n} c_{a+l,p} e_{a+k,p}$$

$$+ \sum_{\substack{s=1\\ s\neq a+k\\ e_{s,q+l}\notin A}}^{n} d_{s,q+k} e_{s,q+l} + w_{a+k,q+l} e_{a+k,q+l},$$

and $w_{a+k,q+l} \in K$ is 0 if $e_{a+k,q+l} \in A$.

Since $\operatorname{Ann} N \subseteq \operatorname{Ann}(M/N)$, we have that $\operatorname{Ann} N = \operatorname{Ann} N \cap \operatorname{Ann}(M/N)$ and that $\operatorname{Ann} N$ is represented by its action on B_2 as matrices in A with 0's in the last $n-q$ rows and first q columns.

It is easy to show from this that $\operatorname{Ann} N$ is spanned by its matrix units. Hence A is spanned by $\{\hat{e}_{ij};\ e_{ij} \in A/\operatorname{Ann} N\} \cup \{e_{ik} \in \operatorname{Ann} N\}$. Note also that the computations on the preceding pages insure that $(\hat{e}_{ij})_{uv} = 0$ for $1 \leq i,\ j,\ u \leq q$ and $q+1 \leq v \leq n$ such that $\hat{e}_{ij} \in A/\operatorname{Ann} N$ and $e_{uv} \in A$.

Now let R be the subspace of M spanned by $\{m_i\}_{i=1}^{a}$. The above results show that R is a submodule of M, that $\operatorname{Ann}(M/R)$ is spanned by $\{\hat{e}_{ij};\ e_{ij} \in A/\operatorname{Ann} N,\ i \leq a\} \cup \{e_{ik} \in \operatorname{Ann} N,\ i \leq a\}$ and that $A/\operatorname{Ann}(M/R)$ is isomorphic to the subspace of K_n spanned by $\{\bar{e}_{ij};\ e_{ij} \in A/\operatorname{Ann} N,\ i \geq a+1\} \cup \{e_{ik} \in \operatorname{Ann} N,\ i \geq a+1\}$, where $\bar{e}_{ij} = \hat{e}_{ij} - \sum_{1\leq u\leq a,\, 1\leq v\leq n} (\hat{e}_{ij})_{uv} e_{uv}$.

Let

$$\bar{m}_i = m_i + R,\ a+1 \leq i \leq q$$

$$= m'_i + R,\ q+1 \leq i \leq n,$$

where $\{m_i\}_{i=1}^{q} \cup \{m'_i\}_{i=q+1}^{n}$ is the basis B_2 for M, as discussed near the beginning of Case 2. Then $\{\bar{m}_i\}_{i=a+1}^{n}$ is a basis for (M/R) relative to which it affords the above representation of $A/\operatorname{Ann}(M/R)$. Henceforth we will work in $A/\operatorname{Ann}(M/R)$ and identify elements of $A/\operatorname{Ann}(M/R)$ as represented by their action on $\{\bar{m}_i\}_{i=a+1}^{n}$, with their preimages in A, as represented by their action on B_2, modulo span $\{e_{uv}\}_{u\leq a}$.

We now divide Case 2 into two subcases, and obtain a contradiction for each subcase.

Subcase 1: $\exists j,\ l \in [1, n-q]$ such that $e_{a+j,q+l} \in A/\operatorname{Ann}(M/R)$. Then $\{e_{a+i,q+k}\}_{1\leq i,k\leq n-q} \in A/\operatorname{Ann}(M/R)$, as is easily established and in particular, $w = \sum_{i=1}^{n-q} e_{a+i,q+i} \in A/\operatorname{Ann}(M/R)$. But $w \in \mathbf{Z}(A/\operatorname{Ann}(M/R))$, as may be checked, and $w^2 = 0$. This contradicts hypothesis (iii), and hence subcase 1 cannot hold.

Subcase 2: For all $j,\ k \in [1, n-q]$, $e_{a+j,q+l} \notin A/\operatorname{Ann}(M/R)$. For $1 \leq k \leq n-q$ and $2 \leq l \leq n-q$ we have by direct calculation that $\bar{e}_{a+k,a+l} * \bar{e}_{a+1,a+1} = (d_{a+l,q+1} + c_{a+l,q+1}) e_{a+k,q+1}$. If $d_{a+l,q+1} + c_{a+l,q+1} \neq 0$, then $e_{a+k,q+1} \in A/\operatorname{Ann}(M/R)$, contrary to hypothesis. Therefore $\bar{e}_{a+k,a+l} * \bar{e}_{a+1,a+1} = 0$. For $a+1 \leq i,\ j \leq q$,

$j \notin [a+1,\ a+n-q]$, we obtain that $\bar{e}_{ij}\bar{e}_{a+1,a+1} = (d_{j,q+1} + c_{j,q+1})e_{i,q+1}$. Suppose in addition that $i \in [a+1, a+n-q]$ and that $d_{j,q+1} + c_{j,q+1} \neq 0$. Then $e_{i,q+1} \in A/\mathrm{Ann}(M/R)$, a contradiction. Therefore for $a+1 \leq i \leq a+n-q < j \leq q$ we have $\bar{e}_{ij}\bar{e}_{a+1,a+1} = 0$.

Consider $\bar{e}_{a+1,a+1} \cdot \bar{m}_{a+1} = \bar{m}_{a+1}$. For $1 \leq k \leq n-q$ we have by direct calculation that $\bar{e}_{a+k,a+1} \cdot \bar{m}_{a+1} = \bar{m}_{a+k}$. For $1 \leq k \leq n-q$ and $2 \leq l \leq n-q$, $\bar{e}_{a+k,a+l} \cdot \bar{m}_{a+1} = \bar{e}_{a+k,a+l} \cdot (\bar{e}_{a+1,a+1} \cdot \bar{m}_{a+1}) = (\bar{e}_{a+k,a+l} * \bar{e}_{a+1,a+1}) \cdot \bar{m}_{a+1} = 0$. For $a+1 \leq i,\ j \leq q$, $j \notin [a+1,\ a+n-q]$, we obtain that $\bar{e}_{ij} \cdot \bar{m}_{a+1} = 0$. Finally, for $a+1 \leq i \leq q$, $1 \leq k \leq n-q$, we have $e_{i,q+k} \cdot \bar{m}_{a+1} = 0$, assuming $e_{i,q+k} \in A/\mathrm{Ann}(M/R)$.

Now consider

$$\bar{e}_{a+1,a+1} \cdot \bar{m}_{q+1} = \bar{m}_{q+1} + \sum_{\substack{s=a+2 \\ e_{s,q+1} \notin A/\mathrm{Ann}(M/R)}}^{q} d_{s,q+1}\bar{m}_s.$$

Note that $\bar{e}_{a+1,a+1} \cdot (\bar{e}_{a+1,a+1} \cdot \bar{m}_{q+1}) = (\bar{e}_{a+1,a+1} * \bar{e}_{a+1,a+1}) \cdot \bar{m}_{q+1} = \bar{e}_{a+1,a+1} \cdot \bar{m}_{q+1}$. For $2 \leq k \leq n-q$ we have by direct calculation that

$$\bar{e}_{a+k,a+1} \cdot (\bar{e}_{a+1,a+1} \cdot \bar{m}_{q+1}) = w_{a+k,q+1}\bar{m}_{a+k} + \bar{m}_{q+k} + \sum_{\substack{s=a+1 \\ s \neq a+k \\ e_{s,q+1} \notin A/\mathrm{Ann}(M/R)}}^{q} d_{s,q+k}\bar{m}_s.$$

For $1 \leq k \leq n-q$ and $2 \leq l \leq n-q$ we have

$$\bar{e}_{a+k,a+l} \cdot (\bar{e}_{a+1,a+1} \cdot \bar{m}_{q+1}) = (\bar{e}_{a+k,a+l} * \bar{e}_{a+1,a+1}) \cdot \bar{m}_{q+1} = 0.$$

For $a+n-q+1 \leq i,\ j \leq q$ we obtain

$$\begin{aligned} \bar{e}_{ij} \cdot (\bar{e}_{a+1,a+1} \cdot \bar{m}_{q+1}) &= (\bar{e}_{ij} * \bar{e}_{a+1,a+1}) \cdot \bar{m}_{q+1} \\ &= (d_{j,q+1} + c_{j,q+1})e_{i,q+1} \cdot \bar{m}_{q+1} = (d_{j,q+1} + c_{j,q+1})\bar{m}_i, \end{aligned}$$

assuming $e_{i,q+1} \in A/\mathrm{Ann}(M/R)$. For $a+1 \leq i \leq a+n-q < j \leq q$ we have $\bar{e}_{ij} \cdot (\bar{e}_{a+1,a+1} \cdot \bar{m}_{q+1}) = \bar{e}_{ij}\bar{e}_{a+1,a+1} \cdot \bar{m}_{q+1} = 0$. For $a+n-q+1 \leq i \leq q$, $e_{i,q+1} \cdot (\bar{e}_{a+1,a+1} \cdot \bar{m}_{q+1}) = \bar{m}_i$, again assuming that $e_{i,q+1} \in A/\mathrm{Ann}(M/R)$. Finally, for $a+n-q+1 \leq i \leq q$, $2 \leq k \leq n-q$, $e_{i,q+k} \cdot (\bar{e}_{a+1,a+1} \cdot \bar{m}_{q+1}) = 0$, assuming $e_{i,q+k} \in A/\mathrm{Ann}(M/R)$.

Now let $N_1 = A/\mathrm{Ann}(M/R) \cdot \{\bar{m}_{a+1}\}$, $N_2 = A/\mathrm{Ann}(M/R) \cdot \{\bar{m}_{q+1}\}$, $N_3 = A/\mathrm{Ann}(M/R) \cdot \{\bar{m}_{a+1} + \bar{m}_{q+1}\}$ and $S = A/\mathrm{Ann}(M/R) \cdot \{\bar{m}_{a+1}, \bar{m}_{q+1}\}$. Then N_1, N_2, N_3 and S are all submodules of (M/R), and it is easy to show that $S = N_1 + N_2 = N_1 + N_3 = N_2 + N_3$. The above calculations show that

$$\begin{aligned} N_1 &= \mathrm{span}[\{\bar{m}_{a+k}\}_{k=1}^{n-q}], \\ N_2 &= \mathrm{span}\Bigg[\bar{m}_{q+1} + \sum_{\substack{s=a+2 \\ e_{s,q+1} \notin A/\mathrm{Ann}(M/R)}}^{q} d_{s,q+1}\bar{m}_s,\ \{\bar{m}_i\}_{\substack{i=a+n-q+1 \\ e_{s,q+1} \notin A/\mathrm{Ann}(M/R)}}^{q}, \\ &\qquad \Bigg\{w_{a+k,q+1}\bar{m}_{a+k} + \bar{m}_{q+k} + \sum_{\substack{s=a+1 \\ s \neq a+k \\ e_{i,q+1} \in A/\mathrm{Ann}(M/R)}}^{q} d_{s,q+k}\bar{m}_s\Bigg\}_{k=2}^{n-q}\Bigg], \end{aligned}$$

$$N_3 = \operatorname{span}\Bigg[\bar{m}_{a+1} + \bar{m}_{q+1} + \sum_{\substack{s=a+2\\ e_{s,q+1}\notin A/\operatorname{Ann}(M/R)}}^{q} d_{s,q+1}\bar{m}_s,\ \{\bar{m}_i\}^{q}_{\substack{i=a+n-q+1,\\ e_{i,q+1}\in A/\operatorname{Ann}(M/R)}},$$

$$\Bigg\{(w_{a+k,q+1}+1)\bar{m}_{a+k} + \bar{m}_{q+k} + \sum_{\substack{s=a+1\\ s\neq a+k\\ e_{s,q+1}\notin A/\operatorname{Ann}(M/R)}}^{q} d_{s,q+k}\bar{m}_s\Bigg\}^{n-q}_{k=2}\Bigg].$$

Consider now $\bar{m}_i$, $a+n-q+1\le i\le q$, $e_{i,q+1}\in A/\operatorname{Ann}(M/R)$. For $1\le k$, $l\le n-q$, $\bar{e}_{a+k,a+l}\cdot\bar{m}_i=0$. For $a+n-q+1\le h\le q$, $\bar{e}_{hi}\cdot\bar{m}_i=\bar{m}_h$, assuming $\bar{e}_{hi}\in A/\operatorname{Ann}(M/R)$, in which case $e_{h,q+1}\in A/\operatorname{Ann}(M/R)$. For $a+1\le h$, $j\le q$, $j\neq i$, we have $\bar{e}_{hj}\cdot\bar{m}_i=0$. For $a+1\le h\le q$ and $1\le k\le n-q$, $e_{h,q+k}\cdot\bar{m}_i=0$, assuming $e_{h,q+k}\in A/\operatorname{Ann}(M/R)$. Note finally that for $1\le k\le n-q$, $e_{a+k,i}$ cannot be in $A/\operatorname{Ann}(M/R)$, otherwise $e_{a+k,i}e_{i,q+1}=e_{a+k,q+1}\in A/\operatorname{Ann}(M/R)$, a contradiction. It follows from this calculation that if we set

$$T=\operatorname{span}\{m_i\}^{q}_{\substack{i=a+n-q+1\\ e_{i,q+1}\in A/\operatorname{Ann}(M/R)}},$$

then T is a submodule of (M/R).

Now note that $T\subseteq N_2$, $T\subseteq N_3$, so that $T\subseteq N_2+N_3=S$. Thus, $(N_1+T)+N_2=(N_1+T)+N_3=N_2+N_3=S$. Note also that $(N_1+T)\cap N_2=(N_1+T)\cap N_3=N_2\cap N_3=T$.

In conclusion, the lattice of submodules of M/R has the following sublattice:

$$\begin{tikzcd}[arrows=-] & S \arrow[dl] \arrow[d] \arrow[dr] & \\ N_1+T \arrow[dr] & N_2 \arrow[d] & N_3 \arrow[dl] \\ & T & \end{tikzcd}$$

But this implies that the lattice of submodules of M/R is *not* distributive, which in turn implies that the lattice of submodules of M is not distributive. From this contradiction it follows that Subcase 2 cannot hold, and hence that Case 2 cannot hold. Thus, only Case 1 is possible, and we always have that $A\approx I(Q)$, Q a finite quasi-ordered set.

We now develop some technical results needed for generalizing Theorem 3.1 to a characterization of incidence algebras of lower finite quasi-ordered sets.

Notation. If Λ is a directed set and $\{Y_i\}_{i\in\Lambda}$ is a direct system of sets indexed by Λ, we shall let $\varinjlim\{Y_i\}_{i\in\Lambda}$ denote the direct limit. If $\{Z_i\}_{i\in\Lambda}$ is an inverse system of sets indexed by Λ, we shall let $\varprojlim\{Z_i\}_{i\in\Lambda}$ denote the inverse limit.

Suppose now that M is a module over an algebra A. Let Λ_M be the collection of finite-dimensional submodules of M, directed by inclusion. For $M_j\subseteq M_k\in\Lambda_M$, let

$\varphi_{jk} : A/\mathrm{Ann}\, M_k \to A/\mathrm{Ann}\, M_j$ be the projection of $A/\mathrm{Ann}\, M_k$ by $\mathrm{Ann}\, M_j/\mathrm{Ann}\, M_k$. Then $\{A/\mathrm{Ann}\, M_i\}_{M_i \in \Lambda_M}$ is an inverse system of sets, as is readily verified.

We state without proof the first technical result. Most of it follows from an exercise in [5, Chapter 3].

Lemma 3.2. *Let A be an algebra over a field K. Assume that A has a faithful unital left module M such that every finitely generated submodule of M is finite-dimensional. Assume further that K and each $A/\mathrm{Ann}\, M_i$ have the discrete topology, that $X_{M_i \in \Lambda_M}\{A/\mathrm{Ann}\, M_i\}$ has the resulting product topology, and that A is complete in the M-topology. Then A and $\varprojlim\{A/\mathrm{Ann}\, M_i\}_{M_i \in \Lambda_M}$ are isomorphic as topological algebras.*

We now specialize this result to incidence algebras. Let Q be a lower finite q.o. set and let Λ_Q be the collection of finite order ideals of Q, directed by inclusion. For $Q_j \subseteq Q_k \in \Lambda_Q$ define $\tilde{\varphi}_{jk} : I(Q_k) \to I(Q_j)$ by

$$\tilde{\varphi}_{jk}(f) = \sum_{(x,y) \in \mathrm{SEG}(Q_j)} f(x, y)\delta_{xy},$$

for $f \in I(Q_j)$. Then $\{I(Q_i)\}_{Q_i \in \Lambda_Q}$ is an inverse system of sets, as is readily verified.

Lemma 3.3. *Let Q be lower finite. Assume that $I(Q)$ has the standard topology, that each $I(Q_j)$ has the discrete topology, and that $X\{I(Q_i)\}$ has the resulting product topology. Then $I(Q)$ and $\varprojlim\{I(Q_i)\}_{Q_i \in \Lambda_Q}$ are isomorphic as topological algebras.*

Proof. Recall from Lemma 2.4 that $I(Q)$ has the faithful module $M(Q) = \mathrm{span}\{x\}_{x \in Q}$. It follows from Lemma 2.4 and Theorems 2.8 and 2.16 that every finitely generated submodule of $M(Q)$ is finite-dimensional.

For each $Q_i \in \Lambda_Q$ set $M_i = \mathrm{span}\{x_i\}_{x_i \in Q_i}$. It is easy to see that M_i is a finite-dimensional submodule of $M(Q)$ with $\mathrm{Ann}\, M_i = C(Q_i)$, regarding $C(Q_i)$ as a submodule of $C(Q)$. Hence $I(Q)/\mathrm{Ann}\, M_i \approx C(Q)^*/C(Q_i)^\perp \approx C(Q_i)^* \approx I(Q_i)$.

Note that the standard topology and the $M(Q)$-topology coincide on $I(Q)$, by Lemma 2.6. In addition, results in Section 1 imply that $I(Q)$ is complete in the $M(Q)$-topology. Also, it follows easily from Lemma 2.13 and Theorem 2.16 that the correspondence $Q_i \leftrightarrow M_i$ is a p.o. isomorphism between $\Lambda_{M(Q)}$ and Λ_Q. Finally, for $Q_j \subseteq Q_k$ it is readily verified that the following diagram commutes:

$$\begin{tikzcd} I(Q_k) \arrow[r, "\tilde{\delta}_{jk}"] \arrow[d, "\wr"'] & I(Q_j) \arrow[d, "\wr"] \\ A/\mathrm{Ann}\, M_k \arrow[r, "\varphi_{jk}"'] & A/\mathrm{Ann}\, M_j \end{tikzcd}$$

where φ_{jk} is the map defined prior to Lemma 3.2. Hence we can conclude by Lemma 3.2 that $I(Q)$ and $\varprojlim\{I(Q_i)\}_{Q_i \in \Lambda_Q}$ are isomorphic as topological algebras.

We can now characterize incidence algebras of lower finite quasi-ordered sets.

Theorem 3.4. *Let A be a complete topological algebra over a field K where K has the discrete topology. Then $A \approx I(Q)$, Q a lower finite quasi-ordered set, when $I(Q)$ has the standard topology* $\Leftrightarrow$

(i) *A has a faithful module M with a distributive lattice of submodules. Further, every finitely generated submodule of M is finite-dimensional and A has the M-topology.*

(ii) *For every maximal closed ideal J, $A/J \approx K_n$, for some integer n.*

(iii) *For every closed ideal J, $\mathbf{Z}[A/J]$ is isomorphic to the direct product of copies of K.*

Proof. ($\Rightarrow$) (i) This follows from Lemma 2.4, Theorem 2.8, Theorem 2.16, and Lemma 2.6.

(ii) This is Lemma 1.6.

(iii) This is Lemma 1.7.

($\Leftarrow$) We will show that $A \approx I(O_M)$, where O_M is the lower finite quasi-ordered set described in Definition 2.12.

Let M_i be a finitely-generated, hence finite-dimensional, submodule of M. After appropriate verifications, it follows by Theorem 3.1 that $A/\mathrm{Ann}\, M_i \approx I(Q')$, Q' a finite quasi-ordered set. Now since M_i is a faithful distributive module over $I(Q')$, we obtain that $Q' \approx O_{M_i}$, by Theorem 2.16, so that $A/\mathrm{Ann}\, M_i \approx I(Q_{M_i})$.

Now if $M_j \subseteq M_k$ are finitely-generated, hence finite-dimensional, submodules of M, it is readily verified that the following diagram commutes:

$$\begin{array}{ccc} A/\mathrm{Ann}\, M_k & \xrightarrow{\varphi_{jk}} & A/\mathrm{Ann}\, M_j \\ \wr\!\downarrow & & \downarrow\!\wr \\ I(O_{M_k}) & \xrightarrow[\tilde{\varphi}_{jk}]{} & I(O_{M_j}) \end{array}$$

where φ_{jk} and $\tilde{\varphi}_{jk}$ are the maps defined prior to Lemmas 3.2 and 3.3 respectively.

Note that

$$\{O_{M_i};\ M_i \in \Lambda_M\} = \{Q_i \in \Lambda_{O_M}\},$$

by Lemma 2.13, so that

$$\varprojlim\{I(O_{M_i})\}_{M_i \in \Lambda_M} = \varprojlim\{I(Q_i)\}_{Q_i \in \Lambda_{O_M}}.$$

Suppose now that both $A/\mathrm{Ann}\, M_i$ and $I(O_{M_i})$ have the discrete topology, for each $M_i \in \Lambda_M$, and that $\varprojlim\{A/\mathrm{Ann}\, M_i\}_{M_i \in \Lambda_M}$ and $\varprojlim\{I(O_{M_i})\}_{M_i \in \Lambda_M}$ have the resulting product topologies. Then $\varprojlim\{A/\mathrm{Ann}\, M_i\}_{M_i \in \Lambda_M}$ isomorphic to $\varprojlim\{I(O_{M_i})\}_{M_i \in \Lambda_M}$ as topological algebras, by commutativity of the above diagram. But $A \approx \varprojlim\{A/\mathrm{Ann}\, M_i\}_{M_i \in \Lambda_M}$ by Lemma 3.2 and $I(O_M) \approx \varprojlim\{I(O_{M_i})\}_{M_i \in \Lambda_M}$, $I(O_M)$ having the standard topology, by Lemma 3.3. Hence $A \approx I(O_M)$, when the latter has the standard topology. Note finally that O_M is lower finite, by Corollary 2.14.

4. Characterization of incidence coalgebras

In this section we dualize the results of Section 3, obtaining a characterization of finite-dimensional incidence coalgebras, which leads in turn to a characterization of incidence coalgebras of lower finite quasi-ordered sets.

We begin by considering coalgebra notions needed to dualize axiom (i) in the characterization of incidence algebras.

Let M be a K-vector space and $\omega : M \to M \otimes C$ be any linear map. Then $\psi_\omega : C^* \otimes M \to M$ is the linear map defined by setting $\psi_\omega(f \otimes m) = \sum_{i=1}^n f(c_i)m_i$, for $f \in C^*$ and $m \in M$ such that $\omega m = \sum_{i=1}^n m_i \otimes c_i$. It is easy to show that (M, ω) is a right C-comodule iff (M, ψ_ω) is a unital left C^*-module. We shall abbreviate $\psi_\omega(f \otimes m)$ by the more conventional $f \cdot m$, for $f \in C^*$, $m \in M$.

Definition 4.1. If (M, ω) is a right C-comodule, set $\operatorname{Coann} M = \bigcap\{D$ subcoalgebra of C; $\omega M \subseteq M \otimes D\}$, which is a subcoalgebra of C. A right C-comodule M is *cofaithful* if $\operatorname{Coann} M = C$.

Proof of the following result is an elementary exercise in linear algebra.

Lemma 4.2. *Let C be a coalgebra and M be a right C-comodule. Then $\omega M \subseteq M \otimes \operatorname{Coann} M$ and $\operatorname{Coann} M$ is the unique minimal subcoalgebra of C with this property. Further, $\operatorname{Coann} M$ is finite-dimensional when M is.*

Corollary 4.3. *Let C and M be as above. Then M is a cofaithful left* $\operatorname{Coann} M$-*comodule.*

Lemma 4.4. *Let C and M be as before. Then* $\operatorname{Coann} M = (\operatorname{Ann}_{C^*} M)^\perp$ *and* $\operatorname{Ann}_{C^*} M = (\operatorname{Coann} M)^\perp$.

Proof. Choose $m \in M$ and assume $\omega m = \sum_{i=1}^n m_i \otimes c_i$, where $\{m_i\}_{i=1}^n$ is a linearly independent subset of M. Then for any $f \in \operatorname{Ann} N$, $f \cdot m = \sum_{i=1}^n f(c_i)m_i = 0$, and hence $f(c_i) = 0$, $1 \leq i \leq n$. Thus $\operatorname{Ann} M(c_i) = 0$, $1 \leq i \leq n$, or $c_i \in (\operatorname{Ann} M)^\perp$. We can thus conclude that $\omega M \subseteq M \otimes (\operatorname{Ann} M)^\perp$. Since $(\operatorname{Ann} M)^\perp$ is a subcoalgebra of C, we have that $\operatorname{Coann} M \subseteq (\operatorname{Ann} M)^\perp$.

Conversely, we know by Lemma 4.2 that $\omega M \subseteq M \otimes \operatorname{Coann} M$, and hence for any $m \in M$, $\omega m = \sum_{i=1}^n m_i \otimes c_i$, where $c_i \in \operatorname{Coann} M$. Thus for any $g \in (\operatorname{Coann} M)^\perp$, $g \cdot m = \sum_{i=1}^n g(c_i)m_i = 0$. Therefore $g \in \operatorname{Ann} M$ and we obtain that $(\operatorname{Coann} M)^\perp \subseteq \operatorname{Ann} M$. Hence $(\operatorname{Coann} M)^{\perp\perp} = \operatorname{Coann} M \supset (\operatorname{Ann} M)^\perp$, and it follows that $\operatorname{Coann} M = (\operatorname{Ann} M)^{\perp}$. Also, $\operatorname{Ann} M \subseteq (\operatorname{Ann} M)^{\perp\perp} = (\operatorname{Coann} M)^{\perp}$, and it follows that $\operatorname{Ann} M = \operatorname{Coann} M^{\perp}$.

Corollary 4.5. *Let M and C be as before. Then M is cofaithful $\Longleftrightarrow$ M is a faithful rational left C^*-module.*

We can now dualize axiom (i) in the characterization of incidence algebras.

Lemma 4.6. *Let Q be a lower finite quasi-ordered set. Then $C(Q)$ has a cofaithful right comodule $M(Q)$ with a distributive lattice of subcomodules. If Q is finite, then $M(Q)$ is finite as well.*

Proof. Let $M(Q)$ be the K-vector space with basis $\{x\}_{x\in Q}$. Define $\omega : M(Q) \to M(Q) \otimes C(Q)$ by

$$\omega x = \sum_{u r x} u \otimes (u, x),$$

for $x \in M(Q)$, where $(u, x) \in \operatorname{SEG}(Q)$. Note that for $f \in C(Q)^* \approx I(Q)$,

$$\psi_\omega(f \otimes x) = \sum_{u r x} f(u, x) u.$$

It then follows that $(M(Q), \psi_\omega)$ is identical to the $I(Q)$-module $M(Q)$ constructed in Lemma 2.4. Hence by Lemma 2.4 and Theorem 2.8, $(M(Q), \psi_\omega)$ is a faithful rational left $C(Q)^*$-module with a distributive lattice of submodules. Hence $(M(Q), \omega)$ is a cofaithful right $C(Q)$-comodule with a distributive lattice of subcomodules. It is immediate that if Q is finite, then $M(Q)$ is finite as well.

It is easy to adapt the above proof to show that for Q a lower finite q.o. set, any faithful distributive module over $I(Q)$ is rational.

We now develop some coalgebra notions needed to dualize axiom (iii) in the characterization of finite-dimensional incidence algebras. See [8, Sections 1.4 and 2.1], for more details.

An element c of a coalgebra C over a field K is *group-like* of $\Delta c = c \otimes c$. It is easy to show that a coalgebra C is spanned by group-like elements iff C^* is isomorphic to the direct product of copies of K. Such a coalgebra is necessarily pointed.

Let V be a subspace of a coalgebra C. V is a *coideal* if $\Delta(V) \subseteq V \otimes C + C \otimes V$ and $\varepsilon(V) = 0$. V is a coideal iff $V^{\perp}$ is a subalgebra of C^*. If V is a coideal, then C/V has a unique coalgebra structure such that the natural projection map from C to C/V is a coalgebra homomorphism. It is easy to show from this that $V^{\perp}$ and $(C/V)^*$ are isomorphic algebras.

Definition 4.7. Let C be a coalgebra. Let $\mathbf{C}(C)$ be the subspace of C spanned by

$$\left\{ \sum_{i=1}^{n} f(c_{i2})c_{i1} - f(c_{i1})c_{i2};\ c \in C,\ f \in C^*,\ \Delta c = \sum_{i=1}^{n} c_{i1} \otimes c_{i2} \right\}.$$

Note that $\mathbf{C}(C)^{\perp} = \mathbf{Z}(C^*)$, which is a subalgebra of C. Thus $\mathbf{C}(C)$ is a coideal of C, and $C/\mathbf{C}(C)$ is a coalgebra, which we shall call the *cocenter* of C and abbreviate Cocent C.

We can now dualize axiom (iii) in the characterization of incidence algebras.

Lemma 4.8. *Let D be a subcoalgebra of a coalgebra C over the field K. Then* Cocent D *is spanned by group-like elements* $\Leftrightarrow$ $\mathbf{Z}(D^*)$ *is isomorphic to the direct product of copies of K.*

Proof. Cocent D is spanned by group-like elements iff $[D/\mathbf{C}(D)]^*$ is isomorphic to the direct product of copies of K. But $[D/\mathbf{C}(D)] \approx \mathbf{C}(D)^{\perp} \approx \mathbf{Z}(D^*)$. The result now follows.

We now characterize finite-dimensional incidence coalgebras.

Theorem 4.9. *Let C a finite-dimensional coalgebra over a field K. Then $C \approx C(Q)$, Q a finite quasi-ordered set* $\Leftrightarrow$

(i) *C has a finite-dimensional cofaithful right comodule with a distributive lattice of subcomodules.*

(ii) *Every simple subcoalgebra of C is isomorphic to K_n^*, for some $n \in \mathbf{Z}^+$.*

(iii) *The cocenter of every subcoalgebra of C is spanned by group-like elements.*

Proof. ($\Rightarrow$) Axiom (i) is Lemma 4.6. To establish (ii), note that if D is a simple subcoalgebra of C, then D^* is a finite-dimensional simple algebra and hence $D^{\perp}$ is a maximal closed ideal. Therefore $D^* \approx I(Q)/D^{\perp} \approx K_n$, for some integer n, by Lemma 1.6. Since $K_n \approx (K_n^*)^*$, it then follows from elementary coalgebra theory [9] that $D \approx K_n^*$. This proves axiom (ii). To establish (iii), note that for D an arbitrary subcoalgebra of $C(Q)$, $\mathbf{Z}(D^*) \approx [I(Q)/D^{\perp}]$, which is isomorphic to the direct product of copies of K, by Lemma 1.7. Thus Cocent D is spanned by group-like elements, by Lemma 4.8. This proves axiom (iii).

($\Leftarrow$) Axiom (i) implies that C^* has a finite-dimensional faithful unital left module with a distributive lattice of submodules. Now note that since C is finite-dimensional, every (simple) homomorphic image of C^* is of the form $C^*/D^{\perp} \approx D^*$, for D some (simple) subcoalgebra of C. Then axiom (ii) and the above noted fact imply that every simple homorphic image of C^* is isomorphic to K_n, for some $n \in \mathbf{Z}^+$. Also, axiom (iii), together with Lemma 4.8 and the above fact, imply that the center of every homomorphic image of C^* is isomorphic to the direct product of copies of K. We can thus conclude by Theorem 3.1 that $C^* \approx I(Q)$, Q a finite quasi-ordered set. Now $I(Q) \approx C(Q^*)$, and we thus may conclude that $C \approx C(Q)$.

We now develop some technical results about coalgebras needed for the characterization of incidence coalgebras of arbitrary lower finite q.o. sets.

Lemma 4.10. *Let C be an arbitrary coalgebra and M be a cofaithful right C-comodule. Then*

(1) $C = \Sigma\{\operatorname{Coann} M_i;\ M_i$ *is a finite-dimensional subcomodule of* $M\}$.

(2) *For every finite-dimensional subcoalgebra D of C there is a finite-dimensional subcomodule N of M such that $D \subseteq \operatorname{Coann} N$.*

Proof. (1) Choose $m \in M$. Let $M' = C^* \cdot m$, the left C^*-submodule of M generated by m. Then M' is a rational submodule of M. Since M' is finitely generated, it follows that M' is finite-dimensional. Hence M' is a finite-dimensional subcomodule of M. Since $m \in M'$, it follows that $M = \Sigma\{M_i;\ M_i$ is a finite-dimensional subcomodule of $M\}$.

Now $\operatorname{Ann}_{C^*} M = \operatorname{Ann}_{C^*} \Sigma\{M_i;\ M_i$ is a finite-dimensional subcomodule of $M\} = \bigcap \operatorname{Ann}_{C^*} M_i = \bigcap (\operatorname{Coann} M_i)^{\perp}$, by Lemma 4.4. Also, $\operatorname{Ann}_{C^*} M = 0$ since M is cofaithful, and $\bigcap (\operatorname{Coann} M_i)^{\perp} = (\Sigma \operatorname{Coann} M_i)^{\perp}$. Thus $(\Sigma \operatorname{Coann} M_i)^{\perp} = 0$ and it follows that $\Sigma \operatorname{Coann} M_i = (\Sigma \operatorname{Coann} M_i)^{\perp\perp} = 0^{\perp} = C$.

(2) Let D be a finite-dimensional subcoalgebra of C. Then by (1) $\exists M_1, \ldots, M_k$ finite-dimensional subcomodules of M such that $D \subseteq \operatorname{Coann} M_1 + \cdots + \operatorname{Coann} M_k$. Then $N = M_1 + \cdots + M_k$ is a finite-dimensional submodule of M and $\operatorname{Coann} N = \operatorname{Coann} M_1 + \cdots + \operatorname{Coann} M_k$, as is easily shown. Hence $D \subseteq \operatorname{Coann} N$.

Corollary 4.11. *Let M be a cofaithful right C-comodule. Then $C \approx \varinjlim\{\operatorname{Coann} M_i;$ M_i is a finite-dimensional subcomodule of $M\}$.*

Proof. We know that C is the direct limit of its finite-dimensional subcoalgebras. Since each such subcoalgebra D is contained in $\operatorname{Coann} M_i$, for some finite-dimensional subcomodule M_i, the result follows.

Lemma 4.12. *Let Q be a lower finite quasi-ordered set. Then $C(Q) \approx \varinjlim\{C(Q_i);\ Q_i$ is a finite order ideal of $Q\}$.*

Proof. Let D be any finite-dimensional subcoalgebra of $C(Q)$. Then D is spanned by some finite order ideal of SEG Q, according to Lemma 1.3. Let $\{x_1, \ldots, x_n\}$ be the collection of all $x_i \in Q$ such that $(x_i, x_i) \in D$ and x_i is maximal in Q with respect to this property.

Let Q' be the lower order ideal of Q generated by $\{x_1, \ldots, x_n\}$. Then Q' is finite, since Q is lower finite, and $D \subseteq C(Q')$. Hence $C(Q) \approx \varinjlim\{D;\ D$ is a finite-dimensional subcoalgebra of $C(Q)\} \approx \varinjlim\{C(Q_i);\ Q_i$ is a finite lower order ideal of $Q\}$.

We can now characterize incidence coalgebras of lower finite quasi-ordered sets. Our result is a direct generalization of Theorem 4.9.

Theorem 4.13. *Let C be a coalgebra over a field K. Then $C \approx C(Q)$, Q a lower finite q.o. set $\Longleftrightarrow$*

(i) *C has a cofaithful right comodule M with a distributive lattice of subcomodules.*

(ii) *Every simple subcoalgebra of C is isomorphic to K_n^*, for some $n \in \mathbf{Z}^+$.*

(iii) *The cocenter of every subcoalgebra of C is spanned by group-like elements.*

Proof. ($\Longrightarrow$) This is identical to proof of ($\Longrightarrow$) in Theorem 4.9.

($\Longleftarrow$) M is a faithful rational left C^*-module with a distributive lattice of submodules, and every finitely generated submodule of M is finite-dimensional. Hence O_M is lower finite, by Corollary 2.14, O_M being the quasi-ordered set described in Definition 2.12. We will show that $C \approx C(O_M)$.

We establish first that $\operatorname{Coann} M_i \approx C(O_{M_i})$, for M_i any finite-dimensional submodule of M. Note that any such M_i has a distributive lattice of submodules, hence of subcomodules. Further, M_i is a finite-dimensional cofaithful right $\operatorname{Coann} M_i$-comodule, by Lemma 4.2 and Corollary 4.3. Hence $\operatorname{Coann} M_i$ is a finite-dimensional coalgebra over K having a finite-dimensional cofaithful right comodule with a distributive lattice of subcomodules, namely M_i. Further, every simple subcoalgebra of $\operatorname{Coann} M_i$ is isomorphic to K_n^*, for some integer n, and the cocenter of every subcoalgebra of $\operatorname{Coann} M_i$ is spanned by group-like elements, these properties being inherited from C. Then by Theorem 4.9, $\operatorname{Coann} M_i \approx C(Q')$, Q' a finite quasi-ordered set, and hence $(\operatorname{Coann} M_i)^* \approx I(Q')$. Since M_i is a faithful distributive module over $I(Q')$, we conclude that $Q' \approx O_{M_i}$, by Theorem 2.16. Hence $\operatorname{Coann} M_i \approx C(O_{M_i})$.

We note second that if M' is an arbitrary subcomodule of M, then M' is finite-dimensional iff $O_{M'}$ is a finite order ideal of O_M, and that every finite order ideal of O_M is of the form $O_{M'}$, M' some finite-dimensional subcomodule of M. This follows from Lemma 2.13.

Finally, $C \approx \varinjlim\{\operatorname{Coann} M_i;\ M_i$ is a finite-dimensional subcomodule of $M\}$, by Corollary 4.11, which is isomorphic to $\varinjlim\{C(O_{M_i})\}$, by first remark, which in turn equals $\varinjlim\{C(Q_i);\ Q_i$ is a finite order ideal of $O_M\}$, by the second remark, which is isomorphic to $C(O_M)$, by Lemma 4.12. Hence $C \approx C(O_M)$.

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